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Charathéodory's "Royal Road" to the Calculus of Variations: A Possible Bridge Between Classical and Quantum Physics

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Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/53843>

1. Introduction

Constantin Carathéodory, a Greek-born, well-known German mathematician, is rarely mentioned in connection to physics. One of his most remarkable contributions to mathematics is his approach to the calculus of variations, the so-called Carathéodory's "royal road" [1]. Among physicists, Carathéodory's name is most frequently related to his contributions to the foundations of thermodynamics [2] and to topics of classical optics, though, as a pupil of Hermann Minkowski, he also worked on the development of especial relativity. In our opinion, however, Caratheodory's formulation of the variational problem deserves to be better known among physicists. For mathematicians, features like rigor and non-redundancy of basic postulates are of utmost importance. Among physicists, a more pragmatic attitude is usually behind efforts towards a theoretical construction, whose principal merit should be to offer an adequate description of Nature. Such a construction must provide us with predictive power. Rigor of the theoretical construction is necessary but not sufficient. Elegance – which is how non-redundancy and simplicity usually manifest themselves – can be sometimes just a welcome feature. Some other times, however, elegance has become a guiding principle when guessing at how Nature works. Nevertheless, once the basic principles of a theoretical construction have been identified, elegance may recede in favor of clarity, and redundancy might become acceptable. Such differences between the perspectives adopted by mathematicians and physicists have been presumably behind the different weight they have assigned to Caratheodory's achievements in the calculus of variations. To be sure, variational calculus does play a central role in physics, nowadays even more than ever before. It is by seeking for the appropriate Lagrangian that we hope to find out the most basic principles ruling physical behavior. Concepts like Feynman's path integral

have become basic tools for the calculation of probability amplitudes of different processes, as well as for suggesting new developments in quantum field theory. Symmetry, such a basic concept underlying those aspects of Nature which appear to us in the form of interactions among fundamental particles, is best accounted for within the framework of a variational principle.

Within the domain of classical physics, only two fundamental interactions have been addressed: the gravitational and the electromagnetic interactions. The theoretical construction may correspondingly rest on two variational principles, one for gravitation and the other for electromagnetism. These principles lead to so-called “equations of motion”: the Maxwell and the Lorentz equations for electromagnetism, and the Einstein and the geodesic equations for gravitation. All these differential equations can be derived as Euler-Lagrange equations from the appropriate Lagrangian or Lagrangian density.

The usual approach to variational calculus in physics starts by considering small variations of a curve which renders extremal the action integral $\int L dt$, with L being the Lagrangian. This leads to the Euler-Lagrange equations of motion. By submitting L to a Legendre transformation one obtains the corresponding Hamiltonian, in terms of which the Hamilton equations of motion can be established. By considering canonical transformations of these equations, one arrives at the Hamilton-Jacobi equation for a scalar function $S(t, x)$. It is last one that has been used to connect the classical approach with the quantum one, e.g., in Madelung’s hydrodynamic model [3] or in Bohm’s “hidden variables” approach [4]. This appears natural, because both the Hamilton-Jacobi and the Schrödinger equation rule the dynamics of quantities like $S(t, x)$ and $\psi(t, x)$, respectively, which are scalar fields. Their scalar nature is in fact irrelevant; they could be tensors and spinors. The relevant issue is that while the Euler-Lagrange and the Hamilton equations refer to a single path, quantum equations address a field. The quantum-classical connection thus requires making a field out of single paths, something which occurs by going to the Hamilton-Jacobi equation, or else by establishing a path-integral formulation, as Feynman did. The latter considers a family of trajectories and assigns a probability to each of them. Now, Carathéodory’s approach has the advantage of addressing right from the start a field of extremals. In fact, as the calculus of variations shows, a solution of the extremal problem exists only when the sought-after extremal curve can be embedded in a field of similar extremals. Carathéodory exploited this fact by introducing the concepts of “equivalent variational problems” and the “complete figure”. It is then possible to elegantly derive from a single statement the Euler-Lagrange and the Hamilton equations, as well as the Hamilton-Jacobi equation, all of them as field equations. The familiar Euler-Lagrange and Hamilton equations can be obtained afterwards by singling out a particular extremal of the field. But – as already stressed – it is not the inherent elegance of the formulation what drives our interest towards Carathéodory’s approach. It is rather its potentiality as a bridge between classical and quantum formulations what should be brought to the fore. Indeed, Carathéodory’s approach can provide new insights into the connection between classical and quantum formulations. These insights could go beyond those already known, which were obtained by extending the Hamilton-Jacobi equation with the inclusion of additional terms. By dealing with the other field equations that appear within Carathéodory’s approach, one may hope to gain additional insight.

The present chapter, after discussing Carathéodory’s approach, shows how one can classically explain two phenomena that have been understood as being exclusively quantum

mechanical: superconductivity and the response of a sample of charged particles to an external magnetic field. The London equations of superconductivity were originally understood as an *ad-hoc* assumption, with quantum mechanics lying at its roots. On the other hand, according to classical mechanics there can be no diamagnetism and no paramagnetism at all. We will deal with these two issues, showing how it is possible to classically derive the London equations and the existence of magnetic moments. This is not to say that there is a classical explanation of these phenomena. What is meant is that, specifically, the London equations of superconductivity can be derived from a classical Lagrangian. It is worth noting that a previous attempt in this direction, due to W. F. Edwards [5], proved false [6–8]. The failure was due to an improper application of the principle of least action. The approach presented here is free from any shortcomings. It leads to the London equations both in the relativistic and in the nonrelativistic domains. It should be stressed that this does not explain the appearance of the superconducting phase. It only shows how the London equations follow from a purely classical approach. Also the expulsion of a magnetic field from the interior of a superconductor, i.e., the Meissner effect, follows. That is, perfect diamagnetism can be explained classically, as has been shown recently [9] but under restricted conditions. This is in contradiction with the Bohr-van Leeuwen theorem, according to which there can be no classical magnetism [10]. This point has been recently discussed (see, e.g., [11]) and it has been shown that the Bohr-van Leeuwen theorem does not hold when one uses the Darwin Hamiltonian, which was proposed back in 1920. The Darwin Hamiltonian contains additional terms with respect to the standard one that is used to describe a charged particle interacting with an electromagnetic field. Applying Carathéodory's approach it can be shown that it is unnecessary to go beyond the standard Hamiltonian or Lagrangian to conclude that a magnetic response may be explained classically. The main point is that the Bohr-van Leeuwen theorem did not consider a constant of the motion which in Carathéodory's approach naturally arises. By considering this constant of the motion, the possibility of magnetic response in a sample of charged particles automatically appears.

After dealing with the above two cases, the rest of the chapter will be devoted to show how gauge invariance can be considered within Carathéodory's framework. This may have some inspiring effect for future work aiming at exploring the quantum-classical correspondence.

2. Carathéodory's royal road

2.1. Preliminaries

Let us begin by recapitulating the approach usually employed in physics. For the sake of describing a particle's motion we use a variational principle based on a Lagrangian L . When describing the dynamics of a field we use instead a variational principle based on a Lagrangian density \mathcal{L} . The Euler-Lagrange equations are, respectively,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad (1)$$

for a Lagrangian $L(t, x^i, \dot{x}^i)$, with $i = 1, \dots, n$, and

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^i)} \right) - \frac{\partial \mathcal{L}}{\partial \psi^i} = 0, \quad (2)$$

for a Lagrangian density $\mathcal{L}(\psi^i, \partial_\mu \psi^i)$ that depends on n fields ψ^i and their derivatives $\partial_\mu \psi^i$ with respect to space-time coordinates x^μ . The convention of summing over repeated indices has been used in Eq.(2), as we will do henceforth. The above equations are necessary conditions that are derivable from the action principle

$$\delta I = 0, \quad (3)$$

with the action given by $I = \int L dt$ for the particle motion and $I = \int \mathcal{L} d^4x$ for the field dynamics. The variation δ means that we consider different paths joining some fixed initial and end points – hypersurfaces in the case of $\int \mathcal{L} d^4x$ – and seek for the path that affords I an extremal value. Curves which are solutions of the Euler-Lagrange equations are called *extremals*.

Let us concentrate on the case $I = \int L dt$ in what follows and sketch how the standard derivation of Eq.(1) is usually obtained: one takes the variation $\delta \int L dt = \int dt \left[(\partial L / \partial x^i) \delta x^i + (\partial L / \partial \dot{x}^i) \delta \dot{x}^i \right]$, and observing that $\delta \dot{x}^i = d(\delta x^i) / dt$, integration by parts gives $\delta I = \int dt \left(\partial L / \partial x^i - d(\partial L / \partial \dot{x}^i) / dt \right) \delta x^i = 0$, where we have considered that $\delta x^i = 0$ at the common endpoints of all the paths involved in the variation. The arbitrariness of δx^i leads to Eq.(1) as a necessary condition for δI to be zero.

The important case of a time-independent Lagrangian ($\partial L / \partial t = 0$) leads to the conservation of the quantity

$$\frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \quad (4)$$

along an extremal, as can be seen by taking its time-derivative and using Eq.(1). By introducing the canonical momenta $p_i = \partial L(t, x, \dot{x}) / \partial \dot{x}^i$ and assuming that we can solve these equations for the \dot{x}^i as functions of the new set of independent variables, $\dot{x}^i = \dot{x}^i(t, x^i, p_i)$, we can define a Hamiltonian $H(t, x, p)$ through the expression given by Eq.(4), written in terms of the new variables (t, x^i, p_i) :

$$H(t, x, p) = p_i \dot{x}^i(t, x, p) - L(t, x, \dot{x}(t, x, p)). \quad (5)$$

The Euler-Lagrange equations are then replaced by the *Hamilton equations*:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \quad (6)$$

Eq.(5) can be seen as a Legendre transformation leading from the set (x^i, \dot{x}^i, t) to the set (x^i, p_i, t) by means of the function $H(t, x, p)$. Taking the differential of $H(t, x, p)$ on the left-hand side of Eq.(5),

$$dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial p_i} dp_i, \quad (7)$$

and on the right-hand side,

$$dH = p_i d\dot{x}^i + \dot{x}^i dp_i - \frac{\partial L}{\partial x^i} dx^i - \frac{\partial L}{\partial \dot{x}^i} d\dot{x}^i - \frac{\partial L}{\partial t} dt, \quad (8)$$

and replacing $\partial L / \partial \dot{x}^i$ by p_i , after equating both sides we see that Eqs.(6) must hold true, together with

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (9)$$

A third way to deal with the motion problem is given by the *Hamilton-Jacobi* equation. In order to introduce it, one usually starts by considering canonical transformations, i.e., those being of the form $\{x^i, p_i\} \rightarrow \{X^i(x, p, t), P_i(x, p, t)\}$ and leaving the action I invariant. They lead to equations for $\{X^i, P_i\}$ that are similar to Eqs.(6) but with a new Hamiltonian, $K(t, X, P)$. From the set of canonical variables $\{x^i, p_i, X^i, P_i\}$ only $2n$ of them are independent. One considers then four types of transformations, in accordance to the chosen set of independent variables: $\{x^i, X^i\}$, $\{x^i, P_i\}$, $\{p_i, X^i\}$, and $\{p_i, P_i\}$. The transformation from the old to the new canonical variables can be afforded by a so-called "generating function" S , which depends on the chosen set of independent variables and the time t . The old and new Hamiltonians are related by $K = H + \partial S / \partial t$. If we succeed in finding a transformation such that $K = 0$, the Hamilton equations for K can be trivially solved. One is thus led to seek for a transformation whose generating function is such that $K = 0$. When the set of independent variables is $\{x^i, P_i\}$ the p_i are given by $p_i = \partial S / \partial x^i$, while the new momenta have constant values $P_i = \alpha_i$ in virtue of $K = 0$. This last equation reads, in terms of the original Hamiltonian,

$$\partial S / \partial t + H(t, x^i, \partial S / \partial x^i) = 0. \quad (10)$$

This is the Hamilton-Jacobi equation. It has played an important role beyond the context in which it originally arose, becoming a sort of bridge that links classical and quantum mechanics. As a first attempt to obtain a quantum-mechanical formalism it was Sommerfeld who, following Bohr, considered action-angle variables for the case of a conservative Hamiltonian, $\partial H / \partial t = 0$. This Hamiltonian was furthermore assumed to allow the splitting of $S(t, x^i, \alpha_i)$ as $S = Et - \sum_j W_j(x^j; \alpha_j)$. Restricting the treatment to cases where the relationship between the p_i and the x^i , given through $p_i = \partial S / \partial x^i$, is such that the orbits $p_i = p_i(x^j, \alpha_j)$ are either closed (*libration-like*) or else periodic (*rotation-like*), action-angle variables can be introduced as new canonical variables [12]. By imposing that the action variables are integer-multiples of a fundamental action, i.e., Planck's h , it was possible to obtain a first formulation of quantum mechanics. This version is known as "old quantum mechanics". A second attempt went along Schrödinger's reinterpretation of the left-hand-side of Eq.(10) as a Lagrangian density of a new variational principle. Schrödinger considered first the case $\partial H / \partial t = 0$, with $H = \sum_{i=1,3} p_i^2 / 2m + V$, and introduced ψ through $S = k \ln \psi$, with k a constant. From the left-hand side of Eq.(10), after multiplying it by ψ^2 , Schrödinger obtained an expression that he took as a Lagrangian density: $\mathcal{L} = \sum_{i=1,3} k^2 (\partial \psi / \partial x^i)^2 / 2m + (V - E) \psi^2$. Inserting this \mathcal{L} into the Euler-Lagrange equations (2) one readily obtains the (time-independent) Schrödinger equation. The constant k could be

identified with \hbar by comparison with Bohr's energy levels in the case of the hydrogen atom ($V \sim 1/r$). We recall that parallel to this approach, another one, due to Heisenberg, Born, Jordan and Dirac, was constructed out of a reformulation of the action-angle formalism applied to multiple periodic motions. This reformulation led to a formalism in which the Poisson brackets were replaced by commutators, and the canonical variables by operators.

Coming back to the general action principle, we have so far followed the road usually employed by physicists. This road was built out of manifold contributions, made at different times and with different purposes. As a consequence, it lacks the unity and compactness that a mathematical theory usually has. At the beginning of the 20th century mathematicians were concerned with the construction and extension of a sound theory for the calculus of variations. It is in this context that Carathéodory made his contributions to the subject. They were thus naturally conceived from a mathematical viewpoint. Apparently, they added nothing new that could be of use for physicists, and so passed almost unnoticed to them. Our purpose here is to show how Carathéodory's formulation can provide physical insight and inspire new approaches. In the following, we give a short account of Carathéodory's approach. We will try to show the conceptual unity and potential usefulness that Carathéodory's formulation entails. Such a unity roots on the so-called *complete figure* that Carathéodory introduces as a central concept of his approach. It serves as the basis of a formulation in which the Euler-Lagrange, the Hamilton and the Hamilton-Jacobi equations appear as three alternative expressions of one and the same underlying concept.

2.2. The non-homogeneous case

Let us first consider the so-called *non-homogeneous* case, i.e., one in which the action principle – and with it the Euler-Lagrange equation – is not invariant under a change of the curve parameter. In physics, this parameter usually corresponds to time. By solving the equations of motion one obtains not only the geometrical path traced by the particle – or group of particles – being described, but also how, i.e., the rate at which this path is traveled. The *non-homogeneous* case applies to non-relativistic formulations.

The equation of motion follows from the variational principle $\delta \int L(t, x^i, \dot{x}^i) dt = 0$. As physicists, we usually visualize the variational principle as expressing how Nature works: among all possible paths joining two given points, Nature chooses the one which affords $\int L dt$ an extremal value. In some sense, this presupposes a non-local behavior, as two distant points determine the extremal curve that should join them. This is reminiscent of the action-at-a-distance invoked by earlier formulations, in whose context the variational principle originally arose. The approach proposed by Carathéodory is more in accordance with our modern view of local interactions. He replaced the problem of finding an extremum for the action integral by one of finding a local extremal value for a function. Thus, the field concept is at the forefront, playing a major role.

Let us recall some important assumptions [1, 13–15] concerning the central problem of variational calculus:

a) To find an extremal curve $x^i = x^i(t)$ that satisfies $\delta \int L dt = 0$ requires that we restrict ourselves to a simply-connected domain. Though apparently technical, this point might entail a profound physical significance.

b) An extremal curve exists only in case that it can be embedded in a whole set of extremals, a so-called "Mayer field".

Now, having a field of curves is equivalent to defining a vector field $v^i(t, x^j)$: at each point x^j we just define $v^i(t, x^j)$ to be tangent to the unique curve which goes through x^j . In other words, the curves that constitute the field are integral curves of $v^i(t, x^j)$:

$$\frac{dx^i(t)}{dt} = v^i(t, x(t)). \quad (11)$$

Finding all the extremals $x^i(t)$ is thus equivalent to fixing $v^i(t, x)$. Once we have $v^i(t, x)$, the extremals can be obtained by integration of Eq.(11). The task of finding $v^i(t, x)$ can be approached locally. To this end, observe that the extremals we are seeking, for which $\delta \int L dt = 0$, are also extremals of the modified, "equivalent variational problem", $\delta \int (L - dS/dt) dt = 0$. This can be written as

$$\delta \int (L - \partial_t S - \dot{x}^i \partial_i S) dt = 0. \quad (12)$$

Now, assume that we are dealing with a particular Lagrangian $L^*(t, x, v)$, for which the following requirements are met: First, it is possible to find a vector field v such that $L^*(t, x, v) = 0$. Second, $L^*(t, x, w) > 0$ for any other field $w \neq v$. It is then easy to show that the integral curves of v are extremals of the variational problem $\delta \int L^* dt = 0$. Of course, not every Lagrangian will satisfy the requirements we have put on L^* ; but by making use of the freedom we have to change our original problem into an "equivalent variational problem", we let $L^* = L - dS/dt$ and seek for a vector field v such that

$$L(t, x, v) - \partial_t S - v^i \partial_i S = 0 \quad (13)$$

identically, the value zero being an extremal one with respect to variations of v . This happens for a suitably chosen $S(t, x)$ that remains fixed in this context. The function $S(t, x)$ must be just the one for which the value of $\int (\partial_t S + \dot{x}^i \partial_i S) dt = \int L(t, x, \dot{x}) dt$, this last integral being calculated along an extremal curve. In other words, among all equivalent variational problems we seek the one for which the conditions put upon L^* are fulfilled. Thus, for the extremal value being, e.g., a minimum, it must hold Eq.(13), while $L(t, x^i, w^i) - \partial_t S - w^i \partial_i S > 0$ for any other field $w \neq v$. In this way our variational problem becomes a local one: v has to be determined so as to afford an extremal value to the expression at the left-hand side of Eq.(13). Thus, taking the partial derivative of this expression with respect to v and equating it to zero we obtain

$$\frac{\partial S}{\partial x^i} = \frac{\partial L(t, x, v(t, x))}{\partial v^i}. \quad (14)$$

Eqs.(13) and (14) are referred to as the *fundamental equations* in Carathéodory's approach. From these two equations we can derive all known results of the calculus of variations. We see, for instance, that defining $p_i = \partial L(t, x, v)/\partial v^i$, Eq.(14) gives $p_i = \partial S(t, x)/\partial x^i$. If

we now introduce, by means of a Legendre transformation, the Hamiltonian $H(t, x, p) = v^i(t, x, p)p_i - L(t, x, v(t, x, p))$, Eq.(13) reads

$$\partial_t S + H(t, x^i, \partial_i S) = 0, \quad (15)$$

which is the Hamilton-Jacobi equation. In this way we obtain an equation for S , the auxiliary function that was so far undetermined. It is also straightforward to deduce the Euler-Lagrange and the Hamilton equations within the present approach. For the sake of brevity, we will show how to derive the Euler-Lagrange equations in the homogeneous case only. The non-homogeneous case can be treated along similar lines.

2.3. The homogeneous case

Let us turn into the so-called homogeneous problem, the one appropriate for a relativistic formulation. In relativity, we consider a space-time continuum described by four variables x^μ . Our variational principle is of the same form as before, i.e., $\delta \int L d\tau = 0$; but we require it to be invariant under Lorentz transformations and under parameter changes. Indeed, all we need in order to fix the motion is the geometrical shape of the extremal curve $x^\mu(\tau)$ in space-time, so that the parameter τ has no physical meaning and the theory must be invariant under arbitrary changes of it. This is achieved when L does not depend explicitly on τ and, furthermore, it is homogeneous of first degree in the generalized velocities \dot{x}^μ : $L(x^\mu, \alpha \dot{x}^\mu) = \alpha L(x^\mu, \dot{x}^\mu)$, for $\alpha \geq 0$. From this requirement, it follows the identity

$$\dot{x}^\mu \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = L, \quad (16)$$

which holds true for homogeneous Lagrangians. This property, however, precludes us from introducing a Hamiltonian in a similar manner as we did in the non-homogeneous case. We come back to this point later on.

As before, we seek also now for a velocity field $v(x)$ and a function $S(x^\mu)$, such that

$$L(x, v) - v^\mu \partial_\mu S = 0, \quad (17)$$

the value zero being an extremal one with respect to v , for a suitably chosen $S(x)$ that remains fixed in this context. For a maximum, for example, it must hold $L(x^\mu, w^\mu) - w^\mu \partial_\mu S < 0$ for any other field¹ $w \neq v$. Differentiating the left-hand side of Eq.(17) with respect to v and equating the result to zero we get

$$\frac{\partial S}{\partial x^\mu} = \frac{\partial L(x, v)}{\partial v^\mu}. \quad (18)$$

¹ The considered fields w^α are essentially different from v^α . A field $w^\alpha = \phi v^\alpha$, with ϕ a scalar function, is essentially the same as v^α .

From the *fundamental equations*, (17) and (18), we can derive all known results also in this case. In particular, we see that $S(x)$ must satisfy the integrability conditions

$$\frac{\partial^2 S}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 S}{\partial x^\nu \partial x^\mu}, \quad (19)$$

which are, as we will shortly see, at the very basis of the Euler-Lagrange equations. Indeed, from Eq.(17) we obtain, by taking the derivative with respect to x^μ ,

$$\frac{\partial L}{\partial x^\mu} + \frac{\partial L}{\partial v^\sigma} \frac{\partial v^\sigma}{\partial x^\mu} = \frac{\partial v^\sigma}{\partial x^\mu} \frac{\partial S}{\partial x^\sigma} + v^\sigma \frac{\partial^2 S}{\partial x^\mu \partial x^\sigma}. \quad (20)$$

On using Eq.(18), Eq.(20) reduces to

$$\frac{\partial L}{\partial x^\mu} = v^\sigma \frac{\partial^2 S}{\partial x^\mu \partial x^\sigma}. \quad (21)$$

From Eqs.(18) and (19) we thus obtain

$$\frac{\partial^2 S}{\partial x^\mu \partial x^\sigma} = \frac{\partial^2 S}{\partial x^\sigma \partial x^\mu} = \frac{\partial^2 L}{\partial x^\sigma \partial v^\mu} + \frac{\partial^2 L}{\partial v^\tau \partial v^\mu} \frac{\partial v^\tau}{\partial x^\sigma}, \quad (22)$$

so that

$$\frac{\partial L}{\partial x^\mu} = v^\sigma \frac{\partial^2 L}{\partial x^\sigma \partial v^\mu} + \frac{\partial^2 L}{\partial v^\tau \partial v^\mu} \frac{\partial v^\tau}{\partial x^\sigma} v^\sigma. \quad (23)$$

If we now evaluate this last relation along a single extremal, $dx^\mu/d\tau = v^\mu(x(\tau))$, we obtain, after recognizing the right hand side of Eq.(23) as $d(\partial L/\partial v^\mu)/d\tau$, the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right). \quad (24)$$

Eq.(23) is therefore more general than the Euler-Lagrange equation. The latter follows from Eq.(23); but not the other way around.

For the non-homogeneous case we obtain a similar result

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad (25)$$

but with the important difference that now the curve-parameter t is fixed: the solution of Eq.(25) provides us not only with the geometrical shape of the extremal curve, but also with the rate at which this curve is traced back.

2.4. The arbitrariness of the curve parameter

Let us see how the arbitrariness of the curve parameter τ manifests itself when dealing with fields of extremals. It is usual to take advantage of such an arbitrariness in order to simplify the equations of motion. It is well known that in the cases of electromagnetism, for which $L(x, \dot{x}) = mc(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} + \frac{e}{c}A_\mu(x)\dot{x}^\mu$, and gravitation, for which $L(x, \dot{x}) = (g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu)^{1/2}$, by choosing τ such that $(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} = 1$, and $(g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu)^{1/2} = 1$, respectively, the equations of motion acquire a simple form. We are so led to ask whether the field v satisfying the fundamental Eqs.(17) and (18) has a corresponding arbitrariness. That this is indeed the case can be seen as follows. We wish to prove that in case v^μ satisfies Eqs.(17) and (18), so does $w^\mu = \phi v^\mu$, with $\phi(x) > 0$ an arbitrary, scalar function. From the homogeneity of the Lagrangian we have $L(x^\mu, \phi v^\mu) = \phi L(x^\mu, v^\mu)$, so that it is seen at once that w^μ satisfies Eq.(17) if v^μ does. Indeed, multiplying Eq.(17) by $\phi(x) > 0$ leads to

$$\phi(x) (L(x, v) - v^\mu \partial_\mu S) = L(x, \phi v) - (\phi v^\mu) \partial_\mu S = L(x, w) - w^\mu \partial_\mu S. \quad (26)$$

The Lagrangian of the “equivalent variational problem” is $L^* = L - v^\mu \partial_\mu S$. Clearly, $L^*(x, \phi v) = \phi L^*(x, v)$, and hence it follows that

$$\frac{\partial L^*(x, w)}{\partial v^\mu} = \frac{\partial L^*(x, w)}{\partial w^\nu} \frac{\partial w^\nu}{\partial v^\mu} = \frac{\partial L^*(x, w)}{\partial w^\mu} \phi. \quad (27)$$

On the other hand,

$$\frac{\partial L^*(x, w)}{\partial v^\mu} = \frac{\partial}{\partial v^\mu} (\phi L^*(x, v)) = \phi \frac{\partial L^*(x, v)}{\partial v^\mu} = \phi \left(\frac{\partial L}{\partial v^\mu} - \frac{\partial S}{\partial x^\mu} \right) = 0, \quad (28)$$

on account of Eq.(17). In view of Eq.(27) we have then that $\partial L^*(x, w)/\partial w^\mu = 0$. In summary, Eqs.(17,18) hold with v being replaced by w , so that both velocity fields solve our variational problem for the same $S(x)$. We have thus the freedom to choose ϕ according to our convenience. The integral curves of $v^\mu(x)$ and $w^\mu(x)$ coincide with each other, differing only in their parametrization.

2.5. Hamiltonians

The introduction of a Hamiltonian offers no problem in the non-homogeneous case, where it was defined as $H(x^i, p_i) \equiv \dot{x}^i(t, x, p)p_i - L(t, x, \dot{x}(t, x, p))$, with $p_i \equiv \partial L/\partial \dot{x}^i$; the condition for solving \dot{x}^i in terms of (x^j, p_j) being assumed to be fulfilled: $\det(\partial^2 L/\partial \dot{x}^i \partial \dot{x}^j) \neq 0$. It is then straightforward [1, 13] to obtain

$$\frac{\partial H}{\partial p_i} = \dot{x}^i = \frac{dx^i}{dt}, \quad (29)$$

which constitute half of the Hamilton equations. It is also easy to show that $\partial H/\partial t = -\partial L/\partial t$ and $\partial H/\partial x^i = -\partial L/\partial x^i$. Using this last result together with $p_i = \partial L/\partial \dot{x}^i$ in the

Euler-Lagrange equation, Eq.(25), one gets

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \quad (30)$$

the other half of the Hamilton equations.

In the homogeneous case, as already mentioned, the corresponding expression for H , i.e., $\dot{x}^\mu \partial L / \partial \dot{x}^\mu - L$, vanishes identically by virtue of Eq.(16). It is nonetheless generally possible to introduce a Hamiltonian in a number of ways. Carathéodory's approach leads to an infinite set of Hamiltonians, from which we can choose the most suitable one for the problem at hand. We will not go into the details here, but refer the reader to the standard literature [1, 13] in which this material is discussed at length.

3. Electromagnetism: The London equations and the Bohr-van Leeuwen theorem

3.1. The London equations of superconductivity

As mentioned before, there are only two interactions relevant to classical physics: electromagnetism and gravitation. In electromagnetism, the Lagrangian is given by

$$L(x, \dot{x}) = mc(\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} + \frac{e}{c} A_\mu(x) \dot{x}^\mu. \quad (31)$$

Here, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric tensor and summation over repeated indices from 0 to 3 is understood. The electromagnetic field is given by the four-potential A_μ , whose components are $\phi(t, x^i)$ and $\mathbf{A}(t, x^i)$.

We are now in a position to show how the London equations follow as a *logical* consequence of the relations presented above, when we use Eq.(31). From Eqs.(18) and (19) we obtain, in general,

$$\frac{\partial}{\partial x^\mu} \left(\frac{L(x, v(x))}{\partial v^\nu} \right) - \frac{\partial}{\partial x^\nu} \left(\frac{L(x, v(x))}{\partial v^\mu} \right) = 0. \quad (32)$$

This equation can be used to obtain the relativistic version of the London equations: As stated before, because the Lagrangian is homogeneous of first order in v , this vector field can be chosen so as to satisfy $(v_\mu v^\mu)^{1/2} = 1$ in the region of interest. From Eq.(32) and Eq.(31) we get

$$\frac{\partial v_\nu}{\partial x^\mu} - \frac{\partial v_\mu}{\partial x^\nu} + \frac{e}{mc^2} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) = 0. \quad (33)$$

This condition leads to the London equations, if we go to the non-relativistic limit, $v^2/c^2 \ll 1$. Indeed, after multiplication by ne , with n meaning a uniform particle density, Eq.(33) can be

brought into the form:

$$\frac{\partial j_\nu}{\partial x^\mu} - \frac{\partial j_\mu}{\partial x^\nu} + \frac{ne^2}{mc^2} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) = 0, \quad (34)$$

where $j_\mu \equiv nev_\mu$. In the non-relativistic limit Eq.(34) reduces, for $\mu, \nu = i, k = 1, 2, 3$, to

$$\frac{\partial j_k}{\partial x^i} - \frac{\partial j_i}{\partial x^k} = -\frac{ne^2}{mc} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right), \quad (35)$$

where we have used $\dot{x}^\mu(\tau) = \gamma(1, \mathbf{v}(t)/c)$ with $\gamma \equiv (1 - \mathbf{v}^2/c^2)^{-1/2} \approx 1$. In three-vector notation this equation reads

$$\nabla \times \mathbf{j} = -\frac{ne^2}{mc} \nabla \times \mathbf{A} = -\frac{ne^2}{mc} \mathbf{B}, \quad (36)$$

which is the London equation [16]. Eq. (36), together with the steady-state Maxwell equation, $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j}$, lead to $\nabla^2 \mathbf{B} = (4\pi ne^2/mc^2) \mathbf{B}$, from which the Meissner effect follows. By considering now the case $\mu = 0, \nu = k = 1, 2, 3$ in Eq.(34), we obtain

$$\frac{\partial j_k}{\partial x^0} - \frac{\partial j_0}{\partial x^k} = \frac{ne^2}{mc^2} \left(\frac{\partial A_0}{\partial x^k} - \frac{\partial A_k}{\partial x^0} \right). \quad (37)$$

Multiplying this equation by $-c^2$ and using three-vector notation it reads, with $j_0 = c\rho$,

$$\frac{\partial \mathbf{j}}{\partial t} + c^2 \nabla \rho = \frac{ne^2}{m} \mathbf{E}. \quad (38)$$

This equation was also postulated by the London brothers as part of the phenomenological description of superconductors. It was guessed as a relativistic generalization of the equation that should hold for a perfect conductor. Without the ρ -term (which in our case vanishes due to the assumed uniformity of n) it is nothing but the Newton, or "acceleration" equation for charges moving under the force $e\mathbf{E}$. The ρ -term was originally conceived as a relativistic "time-like supplement" to the current \mathbf{j} [16]. We see that the London equations are in fact the non-relativistic limit of an integrability condition, Eq.(33), which follows from the variational principle $\delta \int L ds = 0$ alone. The physical content of this procedure appears when we interpret the integral curves of $v(x)$ as streamlines of an ideal fluid. By contracting Eq.(33) with v^μ and using $v^\mu \partial_\nu v_\mu = 0$ (which follows from $v^\mu v_\mu = 1$) we obtain

$$v^\mu \frac{\partial v_\nu}{\partial x^\mu} = \frac{e}{mc^2} F_{\nu\mu} v^\mu, \quad (39)$$

with $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, which relates to \mathbf{E} and \mathbf{B} by $E_i = F_{0i}$ and $B_i = -\epsilon_{ijk}F_{jk}/2$, with ϵ_{ijk} the totally antisymmetric symbol and latin indices running from 1 to 3. The nonrelativistic limit of Eq.(39) reads

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{e}{m} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right). \quad (40)$$

The left-hand side of this equation is the *convective derivative*, which reduces to $d\mathbf{v}/dt$ by restriction to a single extremal. Analogously, Eq.(39) becomes the well-known Lorentz equation when evaluated along a single extremal: $dx^\mu/ds = v^\mu(x(s))$. In this case, $v^\mu(x(s))\partial v_\nu(x(s))/\partial x^\mu = dv_\nu(s)/ds$. Thus, we see that the Lorentz equation for a single particle follows from the more general Eq.(39). For $\mu = 0$ Eq.(39) gives an equation which can be derived from Eq.(40) by scalar multiplication with \mathbf{v} . This is the energy equation. It is worth mentioning that this last fact is a particular manifestation of a well-known result valid for *homogeneous* Lagrangians: only $n - 1$ out of the n Euler-Lagrange equations are independent from each other in this case, due to the identity $\dot{x}^\mu E_\mu = 0$, with $E_\mu \equiv d(\partial L/\partial \dot{x}^\mu)/d\tau - \partial L/\partial x^\mu$ being the Euler vector[13]. Such a result follows from Eq.(16).

Some remarks are in place here. Our derivation of the London equations brings into evidence that they have a validity that goes beyond their original scope. They cannot be seen by themselves as characterizing the phenomenon of superconductivity. Instead, they describe a "dust" of charged particles moving along the extremals of the Lagrangian given by Eq.(31). The field A^μ under which these particles move could be produced by external sources, or else be the field resulting from the superposition of some external fields with those produced by the charges themselves. In this last case, the Maxwell and London equations constitute a self-consistent system. Only under special circumstances, the system of charges can be in a state of collective motion that may be described by the field of extremals obeying Eq.(39). This is the superconducting phase, for which quantum aspects are known to play a fundamental role[17]. However, *once* the phase transition from the normal to the superconducting state has occurred, it becomes possible to describe some aspects of the superconducting state by classical means. This is a case analogous to the one encountered in laser theory. Indeed, several features of a lasing system can be understood within a semi-classical laser theory, whereby the electromagnetic field is treated as a classical, non-quantized field. Perhaps some plasmas could reach the limit of perfect conductivity. However, in order to produce a Meissner-like effect some conditions should be met. It is necessary, for instance, that the available free-energy of the plasma is sufficient to overcome the magnetic field energy, so that the magnetic field can be driven out of the plasma [5–8]. The so-called helicity of the system should also play a role, attaining the value zero for the superconducting state to be reached [9].

In any case, we see that Carathéodory's approach can be a fruitful one in physics. In the case of superconductivity, from the sole assumption that the Lagrangian be given by Eq.(31) one can derive all the equations that were more or less guessed, in the course of almost twenty five years, since Kamerlingh Onnes discovered superconductivity in 1911, until the London model was proposed, in 1935. But beyond this, there are other aspects that can be illuminated by following Carathéodory's approach, as we shall see next.

3.2. Beyond the London equations

Let us address the case when the charge density ρ is not constant, as previously assumed. There is a close relationship between the norm of our velocity field, i.e., $\phi(x) = (v_\mu v^\mu)^{1/2}$, and $\rho(x)$. It can be shown that it is always possible to choose v^μ so that the continuity equation $\partial_\mu j^\mu = 0$ holds. Here, $j^\mu := \rho v^\mu$ and $\rho = nec\phi^{-1}$, n being a free parameter whose dimensions are 1/volume. Indeed, in view of the aforementioned possibility of changing the field v^μ by $w^\mu = \tilde{\phi} v^\mu$, we can always satisfy the continuity equation. For, if $\partial_\mu j^\mu = -f$, we may choose $\tilde{j}^\mu = \tilde{\phi} j^\mu$ such that $\partial_\mu \tilde{j}^\mu = \tilde{\phi} \partial_\mu j^\mu + j^\mu \partial_\mu \tilde{\phi} = 0$. Putting $\psi = \log \tilde{\phi}$, we need to solve $j^\mu \partial_\mu \psi = f$, which is always possible.

Coming back to our Lagrangian of Eq.(31), by replacing it in Eq.(18), we obtain

$$v_\mu = \frac{\phi}{mc} \left(\partial_\mu S - \frac{e}{c} A_\mu \right), \quad (41)$$

with $\phi := (v_\mu v^\mu)^{1/2}$. Using the gauge freedom of A_μ we may replace this field by

$$A'_\mu = A_\mu - \frac{c}{e} \partial_\mu S, \quad (42)$$

in which case Eq.(41) reads

$$v_\mu = -\phi \left(\frac{e}{mc^2} \right) A'_\mu. \quad (43)$$

From this equation and $v_\mu v^\mu = \phi^2$ we get

$$A'_\mu A'^\mu = \left(\frac{mc^2}{e} \right)^2. \quad (44)$$

Eq.(32) applied to the present case gives

$$\frac{\partial}{\partial x^\mu} \left(\frac{v_\nu}{\phi} \right) - \frac{\partial}{\partial x^\nu} \left(\frac{v_\mu}{\phi} \right) + \frac{e}{mc^2} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) = 0. \quad (45)$$

It is clear that this equation holds for A'_μ as well. Eq.(43) is a particular solution of this equation. By Fourier-transforming Eq.(45) we obtain, with $w^\mu := v^\mu / \phi$,

$$k^\mu w^\nu - k^\nu w^\mu = -\frac{e}{mc^2} (k^\mu A^\nu - k^\nu A^\mu). \quad (46)$$

As for the Fourier-transformed version of Eq.(43), it is given by

$$w_\mu(k) = -\frac{e}{mc^2} A'_\mu(k). \quad (47)$$

As we saw before, v^μ can be chosen so that $j_\mu = necw_\mu = (nec/\phi)v_\mu \equiv \rho(x)v_\mu$ satisfies the continuity equation $\partial_\mu j^\mu = 0$. The factor nec is included for dimensional purposes: c/ϕ has no dimensions and n is a free parameter such that ne has dimensions of charge per unit volume. While n is a constant, $\rho(x)$ is a non-uniform charge density. Thus, the scalar field $\phi = (v_\mu v^\mu)^{1/2}$, the norm of the velocity field, is related to the density $\rho(x)$ by $\rho = nec\phi^{-1}$.

Note that $\partial_\mu j^\mu = 0$ implies a restriction on $\partial_\mu v^\mu$. To see this, observe that $\partial_\mu j^\mu = \rho \partial_\mu v^\mu + v^\mu \partial_\mu \rho = 0$. This can be rewritten as

$$\frac{v^\mu}{\phi} \partial_\mu \phi = f, \quad (48)$$

with $\partial_\mu v^\mu = f$. On the other hand, from $v_\mu v^\mu = \phi^2$ it follows that $\phi \partial_\mu \phi = v^\nu \partial_\mu v_\nu$. Eq.(48) then implies that

$$\frac{v^\mu v^\nu}{2\phi^2} (\partial_\mu v_\nu + \partial_\nu v_\mu) = \partial_\sigma v^\sigma. \quad (49)$$

It is also worth noting that instead of Eq.(39) we have now

$$v^\mu \frac{\partial v_\nu}{\partial x^\mu} = \frac{e\phi}{mc^2} F_{\nu\mu} v^\mu + \left(v^\mu \frac{\partial}{\partial x^\mu} \ln \phi \right) v_\nu = \frac{e\phi}{mc^2} F_{\nu\mu} v^\mu + f v_\nu. \quad (50)$$

We could argue that the second term on the right hand side is not physical, because we could choose $\phi = 1$, as we did before, getting Eq.(39). However, such a choice is not available any longer when we invoke charge (or matter) conservation. In such a case, $\partial_\mu j^\mu = 0$, and we must relate j^μ with v^μ by $j^\mu = \rho v^\mu$, so that the above considerations apply.

Coming back to Eq.(47), we see that it implies that $\partial_\mu A'^\mu = 0$, i.e., A'^μ is in the Lorentz gauge. Because of $A'_\mu = A_\mu - (c/e) \partial_\mu S$, the scalar function S must satisfy

$$\partial_\mu \partial^\mu S \equiv \square S = \frac{e}{c} \partial_\mu A'^\mu. \quad (51)$$

Let us consider now Maxwell equations, $\partial_\mu F^{\mu\nu} = (4\pi/c)j^\nu$. If we take j^ν to be the same as before, we are assuming that $F^{\mu\nu}$ is generated by the same currents upon which this field is acting. That is, we are considering a closed system of charges and fields. We have then, using $F^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$ and $\partial_\mu A'^\mu = 0$,

$$\partial_\mu F^{\mu\nu} = \square A'^\nu = \frac{4\pi}{c} j^\nu, \quad (52)$$

while from Eq. (47) we get

$$A'^\nu = -\frac{mc}{ne^2} j^\nu, \quad (53)$$

so that we can write Eq. (52) as

$$\square j^\nu = -\frac{4\pi ne^2}{mc^2} j^\nu \equiv -\frac{1}{\lambda_L^2} j^\nu, \quad (54)$$

in which we have identified the London penetration length λ_L . This equation can be rewritten in the form of the Klein-Gordon equation:

$$(\square + \lambda_L^{-2}) j^\nu = 0, \quad (55)$$

with λ_L replacing $\lambda_C = \hbar/mc$, the Compton wavelength that appears in the Klein-Gordon equation. For the steady-state ($\partial_0 j^\nu = 0$), Eq.(54) reads

$$\nabla^2 j^\nu = +\frac{1}{\lambda_L^2} j^\nu. \quad (56)$$

Taking the usual configuration of a superconductor filling half the space ($z > 0$), the solution of this equation (satisfying appropriate boundary conditions: $\lim_{z \rightarrow \infty} j^\nu(z) = 0$) is

$$j^\nu(z) = \exp\left(-\frac{z}{\lambda_L}\right) j^\nu(0). \quad (57)$$

In general, however, Eq.(55) admits several other solutions that depend on the assumed boundary conditions. Note that Eq.(55) corresponds to a field-free case of the Klein-Gordon equation. This is because $j^\nu \sim A'^\nu$, so that electromagnetic fields and current share the same dynamics. This is a consequence of having assumed that the Euler-Lagrange equations (written as field equations) and Maxwell equations conform a closed system. Notably, A'^ν behaves like a source-free Proca field [18] whose mass (in units of inverse length) is fixed by λ_L .

3.3. The Bohr-van Leeuwen theorem

Dropping the prime, Eq.(53) gives $j^\nu = -(ne^2/mc)A^\nu$, which can be rewritten as $v^\nu = -(e/mc^2)A^\nu$, with $v_\nu v^\nu = 1$. We get thus $v_\nu A^\nu = -mc^2/e$, which in the nonrelativistic limit reads

$$\mathbf{v} \cdot \mathbf{A} = \frac{mc^3}{e}. \quad (58)$$

This condition is important for the following reason. Our considerations have confirmed the possibility of classical diamagnetism, in contradiction with the Bohr-van Leeuwen theorem. Therefore, this theorem should be modified. Eq.(58) represents a constant of the motion that must be taken into account when constructing the phase density for a system of charged particles. The original version of the Bohr-van Leeuwen theorem did not consider condition (58). We will show next how this condition modifies the theorem.

The Bohr-van Leeuwen theorem addresses a sample of charged particles subjected to a uniform magnetic field \mathbf{B} . The nonrelativistic Lagrangian of the system is $L = \sum_{i=1}^N (m_i/2) \mathbf{v}_i^2 + (e_i/c) \mathbf{v}_i \cdot \mathbf{A}$. We can take $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$. The partition function is given by

$$Z = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \left[-\beta \left(\sum_i \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L \right) \right] \left| \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right| d^N \tau, \quad (59)$$

with $d^N \tau$ a properly normalized volume element in configuration space. We see that the terms in L that depend on magnetic potentials are linear in the velocities, so that the integrand in Z turns out to be independent of magnetic potentials. The Bohr-van Leeuwen theorem then follows: because Z is independent of magnetic potentials, there is no effect on the system in response to \mathbf{B} . This prediction changes when we take into account the constant of motion, Eq.(58), or equivalently, $G := (e\mathbf{B}/2mc) \cdot (\mathbf{r} \times \mathbf{v}) = c^2$. For a sample of identical particles we define $G_s := \sum_i (e\mathbf{B}/2mc) \cdot (\mathbf{r}_i \times \mathbf{v}_i) \equiv \sum_i \omega_L \cdot (\mathbf{r}_i \times \mathbf{v}_i)$, with ω_L the Larmor frequency. The phase density D for the corresponding Hamiltonian $H = \sum_i (2m)^{-1} (\mathbf{p}_i - (e_i/c) \mathbf{A}_i)^2$ is given by $D = Z^{-1} \exp(-\beta H - \lambda G_s)$, with Z being the partition function that normalizes D and $\beta = (k_B T)^{-1}$. Both λ and β are Lagrange multipliers, introduced to take account of the restrictions imposed by Eq. (58) and the fixed mean energy, respectively. Thus,

$$D = \frac{1}{Z} \exp \left[\sum_i \left(\frac{-\beta (\mathbf{p}_i - (e/c) \mathbf{A}_i)^2}{2m} + \lambda (\omega_L \cdot \mathbf{r}_i \times \mathbf{v}_i) \right) \right]. \quad (60)$$

The single-particle velocity distribution that can be obtained from D is proportional to

$$\exp \left[-\frac{\beta m}{2} \left(\mathbf{v} - \frac{\lambda}{\beta m} \omega_L \times \mathbf{r} \right)^2 + \frac{\lambda^2}{2\beta m} (\omega_L \times \mathbf{r})^2 \right]. \quad (61)$$

This gives the mean velocity at \mathbf{r} . We have thus $\langle \mathbf{v} \rangle = (\lambda/\beta m) \omega_L \times \mathbf{r}$, which determines the value of the Lagrange multiplier as $\lambda = \beta m$. The phase density can finally be written as

$$D = \frac{1}{Z} \exp \left[-\beta \left(\sum_i \frac{m}{2} \mathbf{v}_i^2 + \mathbf{B} \cdot \mathbf{M} \right) \right], \quad (62)$$

with $\mathbf{M} \equiv \sum_i (e/2c) \mathbf{r}_i \times \mathbf{v}_i$ naturally arising as the magnetic moment of the system. A magnetic response shows up therefore also classically, contrary to what the original version of the Bohr-van Leeuwen theorem stated. It has been shown before that this theorem does not hold whenever the magnetic field produced by the moving charges is taken into account. Such a field is included in the Darwin Lagrangian [11], which is correct to order $(v/c)^2$. In our case, we do not need to modify the standard Lagrangian.

4. Hamilton-Jacobi equations without Hamiltonian

We have already mentioned that for homogeneous Lagrangians the definition of a Hamiltonian is precluded by the vanishing of $\dot{x}^\mu \partial L / \partial \dot{x}^\mu - L$. It is nonetheless possible to introduce a Hamiltonian in a number of ways. Carathéodory's approach leads to an infinite set of Hamiltonians, from which we can choose the most suitable one for dealing with the problem at hand. Here, we focus on the two Lagrangians of interest to us, given by $L = mc(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} + eA_\mu(x)\dot{x}^\mu/c$ for electromagnetism and

$$L(x, \dot{x}) = (g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu)^{1/2} \quad (63)$$

for gravitation. We will prove that in these two particular cases it is possible to derive the equation which the function $S(x)$ has to satisfy, *without* having to introduce a Hamiltonian.

Let us start with gravitation. From Eq.(63) with v^μ replacing \dot{x}^μ , it follows that

$$\frac{\partial L}{\partial v^\mu} = \frac{1}{L} g_{\mu\nu} v^\nu. \quad (64)$$

Using $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$ this equation leads to $v^\nu = Lg^{\mu\nu}p_\mu$, with $p_\mu \equiv \partial L / \partial v^\mu$. Considering that $L^2 = g_{\mu\nu}v^\mu v^\nu = L^2 g^{\mu\nu}p_\mu p_\nu$, it follows $g^{\mu\nu}p_\mu p_\nu = 1$. And because $\partial S / \partial x^\mu = \partial L / \partial v^\mu = p_\mu$, we obtain the Hamilton-Jacobi equation for S :

$$g^{\mu\nu}(x) \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 1. \quad (65)$$

In the electromagnetic case the corresponding Lagrangian leads, by the same token, to $v_\mu = (\partial_\mu S - \frac{e}{c}A_\mu)\phi/mc$ with $\phi \equiv (\eta_{\mu\nu}v^\mu v^\nu)^{1/2}$, again as a consequence of $\partial S / \partial x^\mu = \partial L / \partial v^\mu$. By replacing the above expression of v_μ in $\phi^2 = \eta^{\mu\nu}v_\mu v_\nu$, it follows the Hamilton-Jacobi equation

$$\eta^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} - \frac{e}{c}A_\mu \right) \left(\frac{\partial S}{\partial x^\nu} - \frac{e}{c}A_\nu \right) = m^2 c^2. \quad (66)$$

We remark that there was no need to choose v so as to satisfy either $\phi = \text{const.}$ in the electromagnetic case, or $L = \text{const.}$ in the gravitational case, as it is usually done for obtaining the respective Euler-Lagrange equations in their simplest forms. As a consequence, the constants appearing on the right-hand sides of Eqs.(65) and (66) are independent of the way by which we decide to fix the parameter τ of the extremal curves. Let us remark that it is not unusual to find in textbooks Eq.(65) written with $m^2 c^2$ instead of the 1 on the right-hand side (see, e.g. [19]). This occurs because Eq.(65) is usually introduced as a generalization of Eq.(66), with $A_\mu = 0$ (field-free case). Invoking the equivalence principle, one replaces $\eta^{\mu\nu}$ by $g^{\mu\nu}$ and so arrives at the equation which is supposed to describe a "free" particle moving in a curved space-time region. Now, the metric tensor $g^{\mu\nu}$ embodies all the information that determines how a test particle moves under gravity, irrespective of its inertial mass m . There

is therefore no physical reason to put a term like $m^2 c^2$ on the right-hand side of Eq.(65). To be sure, for all practical purposes it is irrelevant that we set *any* constant on the right-hand side of Eq.(65), as this constant will drop afterwards in the equations describing the motion. But, as a matter of principle, the mass of a test particle should not appear in an equation which describes how it moves under the sole action of gravity.

5. Gauge invariance in electromagnetism and gravitation

Gauge invariance is presently understood as a key principle that lies at the root of fundamental interactions. An equation like Schrödinger's (or Dirac's) for a free electron is invariant under the transformation $\psi \rightarrow \exp(i\alpha)\psi$, for constant α . This is in accordance with the physical meaning of the wave-function and the way it enters in all expressions related to measurable quantities. However, one expects that Nature should respect such an invariance not only *globally*, i.e., with constant α , but also *locally*, with α a function of time and position. It is, so to say, by recourse to the appropriate interaction that Nature manages to reach this goal. For achieving invariance under the $U(1)$ transformation $\psi \rightarrow \exp(i\alpha)\psi$, it is necessary to introduce a *gauge field*, in this case a field represented by $A_\mu(x)$, which couples to the particle. The equation for a free particle is correspondingly changed into one in which A_μ appears. In this context, gauge invariance means invariance under the simultaneous change $\psi \rightarrow \exp(i\alpha)\psi$ and an appropriate one for A_μ . This last one must be so designed that the equation now containing A_μ remains invariant. The change of A_μ turns out to be $A_\mu \rightarrow A_\mu - (\hbar c/e)\partial_\mu\alpha$, which is the one corresponding to a gauge transformation of the *electromagnetic* field. Hence, one is led to interpret electromagnetic interactions as a consequence of local $U(1)$ -invariance. Other fundamental interactions stem from similar gauge invariances: $SU(2) \times U(1)$ gives rise to electroweak interactions, $SU(3)$ to the strong interaction [20], and local Lorentz invariance to gravitation [21, 22].

In this Section we want to show how gauge invariance leads, within the classical context, to considerations paralleling those of quantum mechanics. Carathéodory's formulation will be particularly useful to this end. Let us start with the electromagnetic case. Replacing the Lagrangian of Eq.(31) in the fundamental Eq.(17), we get

$$mc(v_\mu v^\mu)^{1/2} + \frac{e}{c} A_\mu v^\mu - v^\mu \partial_\mu S = 0. \quad (67)$$

Now, the observable predictions we can make concern the integral curves of the velocity field v^μ . This field remains invariant under the replacement

$$A_\mu^* = A_\mu - \frac{c}{e} \partial_\mu W, \quad (68)$$

whenever a simultaneous change in S is undertaken. This change is given by $S \rightarrow S^* = S + W$. It leaves Eq.(67) unchanged, for a fixed $v^\mu(x)$. Eq.(66), to which the velocity field $v^\mu(x)$ belongs, is also fulfilled with S^* and A_μ^* . The quantum-mechanical counterpart of this result could have suggested such a conclusion, in view of the relationship $\psi \sim \exp(iS/\hbar)$. Indeed, a change $\psi \rightarrow \psi^* = \exp(i\alpha)\psi$ means that $\psi^* \sim \exp(iS^*/\hbar)$, with $S^* = S + W$, where $W = \hbar\alpha$.

Now, we are naturally led to ask about a similar invariance in the gravitational case. Here, Eq.(17) reads

$$(g_{\mu\nu}v^\mu v^\nu)^{1/2} - v^\mu \partial_\mu S = 0, \quad (69)$$

and we ask how a simultaneous change of $g_{\mu\nu}$ and S might be, in order that v^μ remains fixed and with it the field of extremals. In the present case, it is better to start with Eq.(23) instead of Eq.(69). The reasons will become clear in what follows. Working out Eq.(23) for the present Lagrangian we obtain, after some manipulations,

$$v^\tau \frac{\partial v^\nu}{\partial x^\tau} + g^{\mu\nu} (\partial_\tau g_{\mu\sigma} - \frac{1}{2} \partial_\mu g_{\sigma\tau}) v^\sigma v^\tau = \frac{\partial(\ln L(x, v(x)))}{\partial x^\tau} v^\nu v^\tau = \frac{\partial\Phi(x)}{\partial x^\tau} v^\nu v^\tau, \quad (70)$$

where $\Phi(x) = \ln L(x, v(x))$. The right-hand side of Eq.(70) can be written in the form $\frac{1}{2}(\delta_\sigma^\nu \partial_\tau \Phi + \delta_\tau^\nu \partial_\sigma \Phi) v^\sigma v^\tau$. This suggests us to symmetrize the coefficient of $v^\sigma v^\tau$ on the left-hand side, thereby obtaining

$$g^{\mu\nu} (\partial_\tau g_{\mu\sigma} - \frac{1}{2} \partial_\mu g_{\sigma\tau}) v^\sigma v^\tau = \frac{1}{2} g^{\mu\nu} (\partial_\tau g_{\mu\sigma} + \partial_\sigma g_{\mu\tau} - \partial_\mu g_{\sigma\tau}) v^\sigma v^\tau \equiv \Gamma_{\sigma\tau}^\nu v^\sigma v^\tau, \quad (71)$$

with $\Gamma_{\sigma\tau}^\nu$ the Christoffel symbols. Eq.(70) then reads

$$v^\tau \frac{\partial v^\nu}{\partial x^\tau} + \Gamma_{\sigma\tau}^\nu v^\sigma v^\tau = \frac{1}{2} (\delta_\sigma^\nu \partial_\tau \Phi + \delta_\tau^\nu \partial_\sigma \Phi) v^\sigma v^\tau. \quad (72)$$

Note that if we choose v such that $L = \text{const.}$, then Eq.(72) becomes the usual geodesic equation, when it is calculated along an extremal curve, $dx^\mu/d\tau = v^\mu(x(\tau))$:

$$\frac{d\dot{x}^\mu}{d\tau} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} = 0. \quad (73)$$

If $L \neq \text{const.}$, we obtain a geodesic equation with a right-hand side of the form $(df/d\tau)\dot{x}^\nu$. In both cases we obtain the same curves – geodesics – but with a different parametrization.

Now, assume that a change, $g_{\mu\sigma} \rightarrow g_{\mu\sigma}^*$ can be found, so that the corresponding $\Gamma_{\tau\sigma}^{*\nu}$ satisfy

$$\Gamma_{\tau\sigma}^{*\nu} - \Gamma_{\tau\sigma}^\nu = \frac{1}{2} (\delta_\sigma^\nu \partial_\tau \Lambda(x) + \delta_\tau^\nu \partial_\sigma \Lambda(x)), \quad (74)$$

with $\Lambda(x)$ being arbitrary. Such a change leads to an equation equivalent to Eq.(70), with Φ being replaced by $\Phi^* = \Phi + \Lambda$, and hence to the same extremals. In this way we recover an old result due to Weyl: if Christoffel symbols are related to each other by Eq.(74), then they have the same geodesics [23]. Given $g_{\mu\nu}$ and Λ , it is always possible to find a $g_{\mu\nu}^*$ satisfying Eq.(74). This is because this equation can be put in the form

$$\partial_\lambda g^{*\mu\nu} = -g^{*\nu\sigma} \Omega_{\lambda\sigma}^\mu - g^{*\mu\sigma} \Omega_{\lambda\sigma}^\nu, \quad (75)$$

with $\Omega_{\lambda\sigma}^\mu := \Gamma_{\lambda\sigma}^\mu + \frac{1}{2}(\delta_\sigma^\mu \partial_\lambda \Lambda + \delta_\lambda^\mu \partial_\sigma \Lambda)$, and it can be straightforwardly proved that the integrability conditions for the above equation are identically satisfied.

In fact, Weyl arrived at a relation like Eq.(74) but having the expression $w_\tau \delta_\sigma^\nu + w_\sigma \delta_\tau^\nu$ on the right hand side, with w_σ taken to be a covariant vector. Now, it is easy to see that w_σ must be a gradient. Indeed, after writing Eq.(74) in Weyl's form, with w_μ replacing $\frac{1}{2}\partial_\mu \Lambda$, we contract both sides of this equation with respect to ν and τ , thereby obtaining $w_\sigma = \frac{1}{5}(\Gamma_{\nu\sigma}^{*\nu} - \Gamma_{\nu\sigma}^\nu)$. Using $\Gamma_{\nu\sigma}^\nu = \frac{1}{2}\partial_\sigma \ln g$, with $g = |\det(g_{\mu\nu})|$, we get $w_\sigma = \partial_\sigma(\ln(g^*/g))/10$.

If we take geodesics as the only observable objects, then it is natural to seek transformations that leave them invariant. Such transformations are given by Eq.(74). However, a transformation of the metric tensor that fulfils Eq.(74) does not leave invariant Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (76)$$

where, we recall, $R_{\mu\nu} = R_{\mu\nu}^\sigma{}_\sigma$ is the Ricci tensor stemming from the Riemann tensor $R_{\mu\nu\sigma}^\lambda$ by contraction of λ and σ , and $T_{\mu\nu}$ means the energy-momentum tensor.

If our transformations do not leave Eq.(76) invariant but we insist in viewing geodesic invariance as a fundamental requirement, then we are led to ask for alternative equations for the gravitational field. These equations should be invariant under Eq.(74). Weyl found a tensor that is invariant under Eq.(74), i.e., a candidate for replacing $R_{\mu\nu\sigma}^\lambda$ as the starting point of the sought-after equations. It is given by

$$W_{\mu\nu\sigma}^\lambda = R_{\mu\nu\sigma}^\lambda - \frac{1}{4}(\delta_\sigma^\lambda R_{\mu\nu} - \delta_\nu^\lambda R_{\mu\sigma}). \quad (77)$$

Unfortunately, any contraction of $W_{\mu\nu\sigma}^\lambda$ vanishes identically, thereby precluding an alternative setting of equations analogous to those of Einstein.

One could argue that it remains still open the possibility of changing our very starting point, so that we should look for a Lagrangian which does not depend on a metric tensor. A natural candidate for this would be an affine connection (the Christoffel symbols being a special case). However, we can show that, even if we start from very general assumptions, we will end up with a Lagrangian like that of Eq. (63). That is, if we take our variational principle in the general form $\delta \int L(x, v) d\tau = 0$, and require that L is invariant under local Lorentz transformations, then L must be of the form $(g_{\mu\nu}(x)v^\mu v^\nu)^{1/2}$. The requirement of invariance under local Lorentz transformations follows from the principle of equivalence: at any given point we can choose our coordinate system so that a body subjected only to gravity appears to move freely in a small neighborhood of the given point. This requirement leads to the particular form of L just given, as can be seen as follows [24]: From the homogeneity of L with respect to v it follows that we can write L in the form $L = (g_{\mu\nu}(x, v)v^\mu v^\nu)^{1/2}$, with $g_{\mu\nu}(x, v) := \frac{1}{2}\partial^2 L^2(x, v)/\partial v^\mu \partial v^\nu$. This puts our variational problem within the framework of Finsler spaces [25]. But local Lorentz invariance implies that $g_{\mu\nu}$ is independent of v , as we shall see, so that we end up within the framework of Riemann spaces, a special case of Finsler spaces.

A transformation in the tangent space, $v \rightarrow w$, defined through $w^\mu = \tilde{\Lambda}_\nu^\mu v^\nu$, is a local Lorentz transformation if it satisfies $g_{\mu\nu}(x, v) = g_{\lambda\sigma}(x, v) \Lambda_\nu^\lambda(x) \Lambda_\nu^\sigma(x)$ at any *fixed* point x . Here, Λ_ν^μ means the inverse of $\tilde{\Lambda}_\nu^\mu$. Invariance of L under local Lorentz transformations means that $L(x^\mu, w^\mu) = L(x^\mu, \tilde{\Lambda}_\nu^\mu v^\nu) = L(x^\mu, v^\mu)$. From this equality, by taking partial derivatives with respect to v , we obtain the two following equations:

$$\frac{\partial L(x, w)}{\partial v^\mu} \Lambda_\nu^\mu = \frac{\partial L(x, v)}{\partial v^\nu} \quad (78)$$

$$\frac{\partial^2 L(x, w)}{\partial v^\mu \partial v^\nu} \Lambda_\sigma^\mu \Lambda_\tau^\nu = \frac{\partial^2 L(x, v)}{\partial v^\sigma \partial v^\tau} \quad (79)$$

When these equations are substituted into the identity

$$g_{\mu\nu}(x, v) = \frac{1}{2} \frac{\partial^2 L^2}{\partial v^\mu \partial v^\nu} = \frac{\partial L}{\partial v^\mu} \frac{\partial L}{\partial v^\nu} + L \frac{\partial^2 L}{\partial v^\mu \partial v^\nu}, \quad (80)$$

one obtains

$$g_{\mu\nu}(x, w) = g_{\lambda\sigma}(x, v) \Lambda_\mu^\lambda(x) \Lambda_\nu^\sigma(x). \quad (81)$$

We conclude therefore, in view of this last equation and the definition of the Lorentz transformation given above, that the equality $g_{\mu\nu}(x, w) = g_{\mu\nu}(x, v)$ holds true for any w and v that are connected to each other by a Lorentz transformation. Thus, setting $w = v + \delta v$, we obtain

$$\frac{\partial g_{\mu\nu}(x, v)}{\partial v^\lambda} = \lim_{\delta v \rightarrow 0} \left(\frac{g_{\mu\nu}(x, v + \delta v) - g_{\mu\nu}(x, v)}{\delta v^\lambda} \right) = 0. \quad (82)$$

Thus, L must be of the form $(g_{\mu\nu}(x) v^\mu v^\nu)^{1/2}$. As we have seen, this result follows from the requirement of local Lorentz invariance. Such an assumption is the counterpart of the condition put by Helmholtz on a general metric space, in order to geometrically characterize Riemann spaces [23]. In this last case, local rotations played the role that is assigned to local Lorentz transformations in the physical case.

6. Summary and conclusions

Carathéodory's approach to the calculus of variations appears to be an appropriate tool for uncovering some aspects of the quantum-classical relationship. Because it describes a whole field of extremals rather than a single one, Carathéodory's approach is, by its very nature, more akin to the quantal formulation. It remains still open how to introduce in this framework the second basic element of the quantal formulation, namely probability. By blending field and probability issues, it is likely that the ensuing result shed some light on questions concerning the quantum-classical correspondence. Here, by way of

illustration of the capabilities of Carathéodory's approach, we have dealt with the two fundamental interactions of classical physics: electromagnetism and gravitation. We have seen that the London equations of superconductivity can be formally derived from the standard Lagrangian of a particle interacting with a prescribed electromagnetic field. The London equations have therefore not a distinctive quantum-mechanical origin, as it is often assumed. This does not mean, however, that we can explain superconductivity by recourse to classical physics alone. The conditions under which a system of charged particles behaves as described by the standard, classical Lagrangian, might be explainable only through quantum mechanics.

In the gravitational case, we recovered Weyl's results about the invariance of geodesics under some special transformation of the Christoffel symbols. Carathéodory's fundamental equations led us to formulate Weyl's result without having to resort to the tools of differential geometry. Furthermore, we have seen that the Lagrangian $L = (g_{\mu\nu}(x)v^\mu v^\nu)^{1/2}$ is a direct consequence of the assumption of local Lorentz invariance. The underlying principle that led us to state the appropriate questions was the principle of gauge invariance, something usually tied to a quantal approach.

In summary, Carathéodory's approach to variational calculus represents an alternative way to introduce some of the most basic principles of classical physics. It unifies different aspects that otherwise appear to be independent from one another, and it can help us in our quest for delimiting the quantum-classical correspondence.

Acknowledgements

Partial financial support from DGI-PUCP and from the Science Department (PUCP) is gratefully acknowledged.

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